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On the uncertainty relations and squeezed states for the quantum mechanics on a circle

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Abstract. The uncertainty relations for the position and momentum of a quantum particle on a circle are identified minimized by the corresponding coherent states. The sqeezed states in the case of the circular motion are introduced and discussed in the context of the uncertainty relations.

PACS numbers: 02.30.Gp, 03.65.-w, 03.65.Sq

1. Introduction

The uncertainty relations are one of the most fundamental concepts of quantum theory. In spite of its importance and long history [1–8] the problem of finding such relations in the case of the quantum mechanics on a circle still remains open. In fact, the experience with the standard Heisenberg uncertainty relations suggests that the uncertainty relations for the quantum mechanics on a circle should be related with the corresponding coherent states. Nevertheless, the existing approaches connecting the uncertainty relations with the coherent states can hardly be called satisfactory.

In this work we introduce the new uncertainties for the position and momentum of a quantum particle on a circle and new uncertainty relations referring to the very recently found coherent states for the circular motion [9,10]. We also introduce the squeezed states for the quantum mechanics on a circle and discuss them in the context of the uncertainty relations.

We begin with a brief account of the alternative approaches linking the uncertainty relations for a quantum particle on a circle to the coherent states. As far as we are aware there are only two such approaches. In the first one we deal with the uncertainty relations implied by the $e(2)$ algebra satisfied by the angular momentum operator and the cosine and sine of the angle operator

$$[\hat{J},\cos\hat{\varphi}] = \mathrm{i}\hbar\sin\hat{\varphi}, \qquad [\hat{J},\sin\hat{\varphi}] = -\mathrm{i}\hbar\cos\hat{\varphi}, \qquad [\sin\hat{\varphi},\cos\hat{\varphi}] = 0. \tag{1.1}$$

These relations are of the form

$$\Delta\hat{J}\Delta\cos\hat{\varphi} \geq \frac{\hbar}{2}|\langle\sin\hat{\varphi}\rangle|, \tag{1.2a}$$

$$\Delta\hat{J}\Delta\sin\hat{\varphi} \geq \frac{\hbar}{2}|\langle\cos\hat{\varphi}\rangle|, \tag{1.2b}$$

$$\Delta\sin\hat{\varphi}\Delta\cos\hat{\varphi} \geq 0. \tag{1.2c}$$

The states minimizing (1.2b) [3] are referred to as the circular squeezed states. Recently, those states have been applied in the study of the Rydberg wave packets [8]. We point out that (1.2a) and (1.2b) cannot be minimized simultaneously [8]. Let us write down the (normalized) wave packets corresponding to the circular squeezed states, i.e., the position representation of these states in the space of square integrable functions on a circle $L^2(S^1)$. We have

$$f_{\alpha,l}(\varphi) = \frac{1}{\sqrt{2\pi I_0(2s)}}\exp[s\cos(\varphi-\alpha)+\mathrm{i}l(\varphi-\alpha)], \tag{1.3}$$

where the packet is peaked at α, $l=\langle\hat{J}\rangle$ is the expectation value of the angular momentum, s is the squeezing and I_0 is a modified Bessel function of the first kind. Of course, the wave packet on a circle should be 2π-periodic. In view of (1.3) this implies that l is integer. But the classical angular momentum is an arbitrary real number. Therefore, the circular squeezed states are not labelled by points of the classical phase space. Bearing in mind that the standard coherent states for a particle on a real line are marked with points of the classical phase space we conclude that (1.3) are rather poor candidate to represent the coherent states for a particle on a circle. On the other hand, it turns out that the uncertainty relations (1.2) cannot be used for determining the correct coherent states for the quantum mechanics on a circle. In our opinion the genuine coherent states for a quantum particle on a circle are those introduced in our joint paper [9] as a solution of some eigenvalue equation (see the next section), and, independently, by Gonzáles and del Olmo [10] who applied the Weil-Brezin-Zak transform. An attempt to connect these coherent states for the quantum mechanics on a circle with the uncertainty relation of the form [6,10]

$$\Delta^2\hat{J}\Delta^2(\hat{\varphi}) \geq \frac{\hbar^2}{4}, \tag{1.4}$$

where

$$\Delta^2(\hat{\varphi}) = \frac{1-|\langle U\rangle|^2}{|\langle U\rangle|^2}, \tag{1.5}$$

where $U=e^{\mathrm{i}\hat{\varphi}}$, was made in [10]. Namely, an upper bond $\hbar$ was found therein for the product of uncertainties $\Delta\hat{J}$ and $\Delta(\hat{\varphi})$ in the (normalized) coherent state $|\xi\rangle$, such that

$$\hbar > \Delta_\xi\hat{J}\Delta_\xi(\hat{\varphi}) > \frac{\hbar}{2}. \tag{1.6}$$

We point out that $\Delta(\hat{\varphi})$ cannot be identified with any uncertainty of the angle. Indeed, in the eigenvector of $\hat{J}$ we have $\Delta\hat{J} = 0$ and $\Delta(\hat{\varphi}) = \infty$. But the maximal uncertainty for the position of a particle on a circle is π, so $\Delta(\hat{\varphi})$ should be taken modulo π. Obviously, $\Delta\hat{J} = 0$ and $\Delta(\hat{\varphi}) \leq \pi$ violate the inequality (1.4), a contradiction. It also seems unlikely that the condition (1.6) allows to determine uniquely the coherent states. It thus appears that the meaning of (1.4) both in the context of the quantum mechanics on a circle and corresponding coherent states is dim. We finally remark that the uncertainty relation (1.4) is implied by (2.5) and the following inequality [11]:

$$\langle A^\dagger A + AA^\dagger\rangle\langle B^\dagger B + BB^\dagger\rangle \geq |\langle A^\dagger B - BA^\dagger\rangle|^2, \tag{1.7}$$

where we set $A = \hat{J} - \langle\hat{J}\rangle$ and $B = U - \langle U\rangle$.

2. Coherent states for the quantum mechanics on a circle

In this section we summarize the elementary facts about the coherent states for a quantum particle on a circle [9,10]. We begin by recalling the basic properties of the quantum mechanics on a circle. Consider a free particle on a circle S^1. The classical Hamiltonian is given by

$$H = \tfrac{1}{2}J^2, \tag{2.1}$$

where J is the angular momentum and we have assumed for simplicity that the particle has unit mass and it moves in a unit circle. Clearly, we have the Poissson bracket of the form

$$\{\varphi, J\} = 1, \tag{2.2}$$

where φ is the angle specifying the position on a circle. The Poisson bracket (2.2) leads according to the rules of the canonical quantization to the commutator

$$[\hat{\varphi}, \hat{J}] = \mathrm{i}, \tag{2.3}$$

where we set $\hbar = 1$. It can be demonstrated that the commutator (2.3) is defined only on the zero vector. Therefore, the better candidate than $\hat{\varphi}$ for representing the position of a quantum particle on a circle is the unitary operator U such that

$$U = e^{i\hat{\varphi}}. \tag{2.4}$$

An immediate consequence of (2.3) and (2.4) is the following algebra:

$$[\hat{J}, U] = U. \tag{2.5}$$

We also point out that (2.5) can be obtained directly from (1.1) and (2.4). Consider the eigenvalue equation

$$\hat{J}|j\rangle = j|j\rangle. \tag{2.6}$$

From (2.5) and (2.6) it follows that the operators U and $U^\dagger$ are the ladder operators. Namely

$$U|j\rangle = |j+1\rangle, \qquad U^\dagger|j\rangle = |j-1\rangle. \tag{2.7}$$

Demanding the time-reversal invariance of the algebra (2.5) we find [9] that the eigenvalues j of the operator $\hat{J}$ can be only integer or half-integer. In this work we restrict, for simplicity, to the case with integer j. We finally write down the orthogonality and completeness conditions satisfied by the vectors $|j\rangle$ such that

$$\langle j|k\rangle = \delta_{jk}, \tag{2.8}$$

$$\sum_{j=-\infty}^{\infty} |j\rangle\langle j| = I. \tag{2.9}$$

We now collect the basic facts about the coherent states for a particle on a circle. These states can be defined by means of the eigenvalue equation [9]

$$Z|\xi\rangle = \xi|\xi\rangle, \tag{2.10}$$

where $Z = e^{-\hat{J}+\frac{1}{2}}U$, and the complex number $\xi = e^{-l+\mathrm{i}\varphi}$ parametrizes the circular cylinder which is the classical phase space for the particle moving in a circle. We remark that in view of the identity

$$Z = e^{\mathrm{i}(\hat{\varphi}+\mathrm{i}\hat{J})}, \tag{2.11}$$

(2.10) has the form analogous to the eigenvalue equation satisfied by the standard coherent states $|z\rangle$ such that

$$e^{ia}|z\rangle = e^{iz}|z\rangle, \tag{2.12}$$

where $a \sim \hat{q} + i\hat{p}$ is the standard Bose annihilation operator and $\hat{q}$ and $\hat{p}$ are the position and momentum observables, respectively. The projection of the vectors $|\xi\rangle$ onto the basis vectors $|j\rangle$ is given by

$$\langle j|\xi\rangle = \xi^{-j}e^{-\frac{j^2}{2}}. \tag{2.13}$$

On using the parameters l and φ (2.13) can be written in the following equivalent form:

$$\langle j|l,\varphi\rangle = e^{lj-\mathrm{i}j\varphi}e^{-\frac{j^2}{2}}, \tag{2.14}$$

where $|l,\varphi\rangle \equiv |\xi\rangle$ with $\xi = e^{-l+\mathrm{i}\varphi}$. The coherent states are not orthogonal. We have

$$\langle\xi|\eta\rangle = \sum_{j=-\infty}^{\infty}(\xi^*\eta)^{-j}e^{-j^2} = \theta_3(\tfrac{\mathrm{i}}{2\pi}\ln\xi^*\eta|\tfrac{\mathrm{i}}{\pi}), \tag{2.15a}$$

$$\langle l,\varphi|h,\psi\rangle = \theta_3(\tfrac{1}{2\pi}(\varphi-\psi) - \tfrac{l+h}{2}\tfrac{\mathrm{i}}{\pi}|\tfrac{\mathrm{i}}{\pi}), \tag{2.15b}$$

where θ_3 is the Jacobi theta-function defined by

$$\theta_3(v|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}(e^{\mathrm{i}\pi v})^{2n}, \tag{2.16}$$

where $q = e^{\mathrm{i}\pi\tau}$ and $\mathrm{Im}\,\tau > 0$. It follows immediately from (2.15) that the squared norm of the coherent states can be written in the form

$$\langle\xi|\xi\rangle = \theta_3(\tfrac{\mathrm{i}}{\pi}\ln|\xi||\tfrac{\mathrm{i}}{\pi}), \tag{2.17a}$$

$$\langle l,\varphi|l,\varphi\rangle = \sum_{j=-\infty}^{\infty} e^{2lj}e^{-j^2} = \theta_3(\tfrac{\mathrm{i}l}{\pi}|\tfrac{\mathrm{i}}{\pi}). \tag{2.17b}$$

The expectation value of the angular momentum $\hat{J}$ in the coherent states obeys

$$\frac{\langle\xi|\hat{J}|\xi\rangle}{\langle\xi|\xi\rangle} \approx l, \tag{2.18}$$

where the maximal error arising in the case $l \to 0$ is of order 0.1% and we have the *exact* equality in the case of l integer or half-integer. Therefore, the parameter l labelling the coherent states can be interpreted as the classical angular momentum. The fact that the parameter φ can be regarded as the classical angle is a consequance of the following formula on the relative expectation value $\langle U\rangle_\xi/\langle U\rangle_1 := \langle\xi|U|\xi\rangle/\langle 1|U|1\rangle$, which is the most natural candidate to describe the average position on a circle:

$$\frac{\langle U\rangle_\xi}{\langle U\rangle_1} \approx e^{\mathrm{i}\phi}, \tag{2.19}$$

where the approximation is very good. More precisely, regardless of the concrete value of l, the maximal error is of order 0.1%. In our opinion the meaning of (2.18) and (2.19) is that the coherent states are as close as possible to the classical phase space.

3. Uncertainty relations for the quantum mechanics on a circle

Our purpose now is to introduce the uncertainties of the momentum and position for a quantum particle on a circle. We first write down the following relation implied by (2.9), (2.17), (2.13) and (2.16):

$$\langle e^{-2\lambda\hat{J}}\rangle_\xi = \frac{\langle\xi|e^{-2\lambda\hat{J}}|\xi\rangle}{\langle\xi|\xi\rangle} = e^{\lambda^2-2l\lambda}\,\frac{\theta_3(l-\lambda|\mathrm{i}\pi)}{\theta_3(l|\mathrm{i}\pi)}. \tag{3.1}$$

On setting $\lambda = \pm 1$ in (3.1) we get

$$\langle e^{-2\hat{J}}\rangle_\xi = e^{1-2l}, \qquad \langle e^{2\hat{J}}\rangle_\xi = e^{1+2l}. \tag{3.2}$$

Further, using (2.9), (2.7), (2.13) and (2.16) we find

$$\langle U^2\rangle_\xi = \frac{\langle\xi|U^2|\xi\rangle}{\langle\xi|\xi\rangle} = e^{-1}e^{2\mathrm{i}\varphi}. \tag{3.3}$$

Eqs. (3.2) and (3.3) taken together yield the remarkable identity

$$\langle e^{-2\hat{J}}\rangle_\xi\langle e^{2\hat{J}}\rangle_\xi = \frac{1}{|\langle U^2\rangle_\xi|^2}. \tag{3.4}$$

We now introduce the following measure of the uncertainty of the angular momentum:

$$\Delta^2_\phi(\hat{J}) := \frac{1}{4}\ln\left(\langle e^{-2\hat{J}}\rangle_\phi\langle e^{2\hat{J}}\rangle_\phi\right) \tag{3.5}$$

and the measure of the uncertainty of the angle

$$\Delta^2_\phi(\hat{\varphi}) := \frac{1}{4}\ln\frac{1}{|\langle U^2\rangle_\phi|^2}, \tag{3.6}$$

where $\langle A\rangle_\phi = \langle\phi|A|\phi\rangle/\langle\phi|\phi\rangle$, so the identity (3.4) can be written as

$$\Delta^2_\xi(\hat{J}) = \Delta^2_\xi(\hat{\varphi}). \tag{3.7}$$

Notice that both uncertainties (3.5) and (3.6) are nonnegative. Indeed, for arbitrary Hermitian operator X we have

$$\langle e^X\rangle\langle e^{-X}\rangle \geq 1, \tag{3.8}$$

following directly from the Schwarz inequality

$$\langle A^\dagger A\rangle\langle B^\dagger B\rangle \geq |\langle A^\dagger B\rangle|^2, \tag{3.9}$$

by putting $A = e^{\frac{X}{2}}$ and $B = e^{-\frac{X}{2}}$. An immediate consequence of (3.8) is nonnegativity of $\Delta^2_\phi(\hat{J})$. The inequality

$$\frac{1}{|\langle U^2\rangle|^2} \geq 1 \tag{3.10}$$

ensuring the positivity of $\Delta^2_\phi(\hat{\varphi})$ is implied by the well-known relation

$$|\langle V\rangle|^2 \leq 1, \tag{3.11}$$

which holds true for arbitrary unitary operator V. The inequality (3.11) can be easily obtained from (3.9) by setting $A = U$ and $B = U^2$.

At first sight the uncertainties (3.5) and (3.6) seem to be weird without any reference to such measures of uncertainties as a standard variance. Nevertheless, we observe that (3.7) has the identical form as the equation satisfied by the variances of the momentum and the position in the standard coherent states for a particle on a real line

$$\Delta^2_z\hat{p} = \Delta^2_z\hat{q}. \tag{3.12}$$

Furthermore, we have the cumulant expansion

$$\langle e^{X}\rangle=\exp(\langle\langle X\rangle\rangle+\tfrac{1}{2!}\langle\langle X^2\rangle\rangle+\tfrac{1}{3!}\langle\langle X^3\rangle\rangle+\tfrac{1}{4!}\langle\langle X^4\rangle\rangle+\ldots),\tag{3.13}$$

where $\langle\langle X^n\rangle\rangle$, $n=1,2,\ldots$, are the cumulants (semiinvariants). The first four cumulants are obtained from moments as

$$\begin{aligned}
\langle\langle X\rangle\rangle &= \langle X\rangle,\\
\langle\langle X^2\rangle\rangle &= \langle X^2\rangle-\langle X\rangle^2,\\
\langle\langle X^3\rangle\rangle &= \langle X^3\rangle-3\langle X^2\rangle\langle X\rangle+2\langle X\rangle^3,\\
\langle\langle X^4\rangle\rangle &= \langle X^4\rangle-4\langle X^3\rangle\langle X\rangle-3\langle X^2\rangle^2+12\langle X^2\rangle\langle X\rangle^2-6\langle X\rangle^4.
\end{aligned}\tag{3.14}$$

Notice that the second cumulant is the usual variance. The third and fourth cumulant is called skewness and curtosis, respectively. Using (3.13), (3.5) and (3.6) we get

$$\Delta^2_\phi(\hat{J}) = \langle\langle\hat{J}^2\rangle\rangle_\phi+\tfrac{1}{3}\langle\langle\hat{J}^4\rangle\rangle_\phi+\tfrac{2}{45}\langle\langle\hat{J}^6\rangle\rangle_\phi+\ldots,\tag{3.15}$$

$$\Delta^2_\phi(\hat{\varphi}) = \langle\langle\hat{\varphi}^2\rangle\rangle_\phi-\tfrac{1}{3}\langle\langle\hat{\varphi}^4\rangle\rangle_\phi+\tfrac{2}{45}\langle\langle\hat{\varphi}^6\rangle\rangle_\phi+\ldots.\tag{3.16}$$

It thus appears that in the first approximation neglecting the cumulants of order four and higher (even) ones, the uncertainties (3.5) and (3.6) coincide with the usual variances of the angular momentum and angle, respectively. We point out that $\Delta^2_\phi(\hat{J})$ vanishes in the eigenstates $|j\rangle$ of $\hat{J}$ when we know the exact value of the angular momentum, and is infinite in the eigenstate $|\varphi\rangle$ of the operator U corresponding to the fixed position on a circle. Analogously, $\Delta^2_\phi(\hat{\varphi})$ vanishes in the state $|\varphi\rangle$ and is infinite in the state $|j\rangle$. We conclude that the uncertainties (3.5) and (3.6) behave correctly in the states with fixed angular momentum and angle. Last but not least we remark that the relations (3.2) and (3.3) take place also in the case with the half-integer eigenvalues of $\hat{J}$.

We now discuss the uncertainty relations for the quantum mechanics on a circle. Equations (3.2)–(3.6) taken together yield

$$\Delta^2_\xi(\hat{J})=\tfrac{1}{2},\qquad \Delta^2_\xi(\hat{\varphi})=\tfrac{1}{2},\tag{3.17}$$

so

$$\Delta^2_\xi(\hat{J})+\Delta^2_\xi(\hat{\varphi})=1.\tag{3.18}$$

The identity (3.18) indicates the following form of the uncertainty relations for a quantum particle on a circle:

$$\Delta^2(\hat{J})+\Delta^2(\hat{\varphi})\geq 1,\tag{3.19}$$

minimized at the coherent states. The uncertainty relations (3.19) are supported by the numerical calculations (see Fig. 1b). We finally point out that (3.19) has the form identical as the uncertainty relations for the sum of variances of the position and momentum of a particle on a real line implied by the standard Heisenberg uncertainty relations, of the form

$$\Delta^2\hat{p}+\Delta^2\hat{q}\geq 1,\tag{3.20}$$

where we set $\hbar=1$.

4. Squeezed states for the quantum mechanics on a circle

We finally study the squeezed states for the quantum mechanics on a circle and the connected uncertainty relations. We first observe that the eigenvectors of the operators $a(s)$ defined as [12]

$$a(s)=e^{-s\frac{\hat{p}^2}{2}}\hat{q}e^{s\frac{\hat{p}^2}{2}}=\hat{q}+is\hat{p},\tag{4.1}$$

where $\hat{q}$ and $\hat{p}$ are the standard position and momentum operators, respectively, and $s>0$ is a real parameter, are the standard squeezed states. In analogy to (4.1) we introduce the operators $Z(s)$ such that

$$Z(s)=e^{-s\frac{\hat{J}^2}{2}}Ue^{s\frac{\hat{J}^2}{2}}=e^{-s(\hat{J}-\frac{1}{2})}U, \tag{4.2}$$

and define the squeezed states $|\xi\rangle_s$ for a quantum particle on a circle by

$$Z(s)|\xi\rangle_s=\xi|\xi\rangle_s. \tag{4.3}$$

It should be noted that in view of (4.2) the coherent states for a particle on a circle satisfying (2.10) correspond to the particular case $s=1$. We also remark that we have a generalization of the formula (2.11) such that

$$Z(s)=e^{\mathrm{i}(\hat{\varphi}+\mathrm{i}s\hat{J})}. \tag{4.4}$$

Therefore, the squeezed states are related with the scaling of the angular momentum. Making use of (4.3), (4.2), (2.6) and (2.7) we easily obtain the following generalizations of the relations (2.12) and (2.13):

$$\begin{aligned}\langle j|\xi\rangle_s &= \xi^{-j}e^{-\frac{sj^2}{2}}, && (4.5a)\\ \langle j|l,\varphi\rangle_s &= e^{lj-\mathrm{i}j\varphi}e^{-\frac{sj^2}{2}}, && (4.5b)\end{aligned}$$

where $|l,\varphi\rangle_s\equiv|\xi\rangle_s$. From (4.5) and (2.9) we derive the overlap integrals such that

$$\begin{aligned}{}_s\langle\xi|\eta\rangle_s &= \sum_{j=-\infty}^{\infty}(\xi^*\eta)^{-j}e^{-sj^2}=\theta_3(\tfrac{\mathrm{i}}{2\pi}\ln\xi^*\eta|\tfrac{\mathrm{i}s}{\pi}), && (4.6a)\\ {}_s\langle l,\varphi|h,\psi\rangle_s &= \theta_3(\tfrac{1}{2\pi}(\varphi-\psi)-\tfrac{l+h}{2}\tfrac{\mathrm{i}}{\pi}|\tfrac{\mathrm{i}s}{\pi}), && (4.6b)\end{aligned}$$

leading to the following expression on the squared norm of the squeezed states:

$$\begin{aligned}{}_s\langle\xi|\xi\rangle_s &= \theta_3(\tfrac{\mathrm{i}}{\pi}\ln|\xi||\tfrac{\mathrm{i}s}{\pi}), && (4.7a)\\ {}_s\langle l,\varphi|l,\varphi\rangle_s &= \sum_{j=-\infty}^{\infty}e^{2lj}e^{-sj^2}=\theta_3(\tfrac{\mathrm{i}l}{\pi}|\tfrac{\mathrm{i}s}{\pi}). && (4.7b)\end{aligned}$$

We point out that the above formulae imply positivity of the parameter s. We also recall that we study the case of the integer eigenvalues of the operator $\hat{J}$.

We now examine the uncertainties in the squeezed states. Taking into account (2.9), (2.6), (4.5) and (4.7) we get a generalization of (3.1)

$$\langle e^{-2\lambda\hat{J}}\rangle_{\xi,s}=\frac{{}_s\langle\xi|e^{-2\lambda\hat{J}}|\xi\rangle_s}{{}_s\langle\xi|\xi\rangle_s}=e^{\frac{\lambda^2}{s}-\frac{2l\lambda}{s}}\,\frac{\theta_3(\frac{l-\lambda}{s}|\frac{\mathrm{i}\pi)}{s}}{\theta_3(\frac{l}{s}|\frac{\mathrm{i}\pi}{s})}. \tag{4.8}$$

Hence, putting $\lambda=\pm s$, we find

$$\langle e^{-2s\hat{J}}\rangle_{\xi,s}=e^{s-2l},\qquad \langle e^{2s\hat{J}}\rangle_{\xi,s}=e^{s+2l}. \tag{4.9}$$

We have also the generalization of (3.3) of the form

$$\langle U^2\rangle_{\xi,s}=\frac{{}_s\langle\xi|U^2|\xi\rangle_s}{{}_s\langle\xi|\xi\rangle_s}=e^{-s}e^{2\mathrm{i}\varphi}. \tag{4.10}$$

By (4.9) and (4.10)

$$\langle e^{-2s\hat{J}}\rangle_{\xi,s}\langle e^{2s\hat{J}}\rangle_{\xi,s}=\frac{1}{|\langle U^2\rangle_{\xi,s}|^2}, \tag{4.11}$$

which leads to the following most natural generalization of the uncertainties (3.5) and (3.6) of the angular momentum and angle, respectively

$$\widetilde{\Delta}^2_{\phi,s_0}(\hat{J}) = \frac{1}{4}\ln\left(\langle e^{-2s_0\hat{J}}\rangle_\phi\langle e^{2s_0\hat{J}}\rangle_\phi\right), \tag{4.12}$$

$$\Delta^2_{\phi,s_0}(\hat{\varphi}) \equiv \Delta^2_\phi(\hat{\varphi}). \tag{4.13}$$

Using the uncertainties (4.12) and (4.13) we arrive at the identity

$$\widetilde{\Delta}^2_{\xi,s_0,s_0}(\hat{J}) = \Delta^2_{\xi,s_0,s_0}(\hat{\varphi}) = \frac{s_0}{2} \tag{4.14}$$

indicating the generalized uncertainty relations such that

$$\widetilde{\Delta}^2_{\phi,s_0}(\hat{J}) + \Delta^2_\phi(\hat{\varphi}) \geq s_0, \tag{4.15}$$

where the equality is reached in the squeezed state $|\xi\rangle_{s_0}$. The uncertainty relations (4.15) are corroborated by the numerical calculations (see Fig. 1 and Fig. 2).

5. Discussion

In this work we have identified the uncertainties and uncertainty relations for the quantum mechanics on a circle minimized by the corresponding coherent states. We have also introduced the squeezed states generalizing the coherent states for a quantum particle on a circle and found the appropriate uncertainty relations saturated by these states. Notice that generalized uncertainty relations (4.15), where the uncertainties are given by (4.12) and (4.13), do not provide any criterion for distinguishing coherent and squeezed states as in the case with the quantum mechanics on a real line. The situation is even more complicated in view of the fact that the squeezed states with different s are not related by a unitary transformation. Namely, we have

$$|\xi\rangle_s = e^{-(s-s_0)\hat{J}^2/2}|\xi\rangle_{s_0}.$$

Thus the states with different s are not unitarily equivalent and the problem naturally arises concerning the physical interpretation of the (dimensionless) parameter s. We point out that in the case of the standard squeezed states for a particle on a real line the states with different squeezing are related by a unitary transformation.

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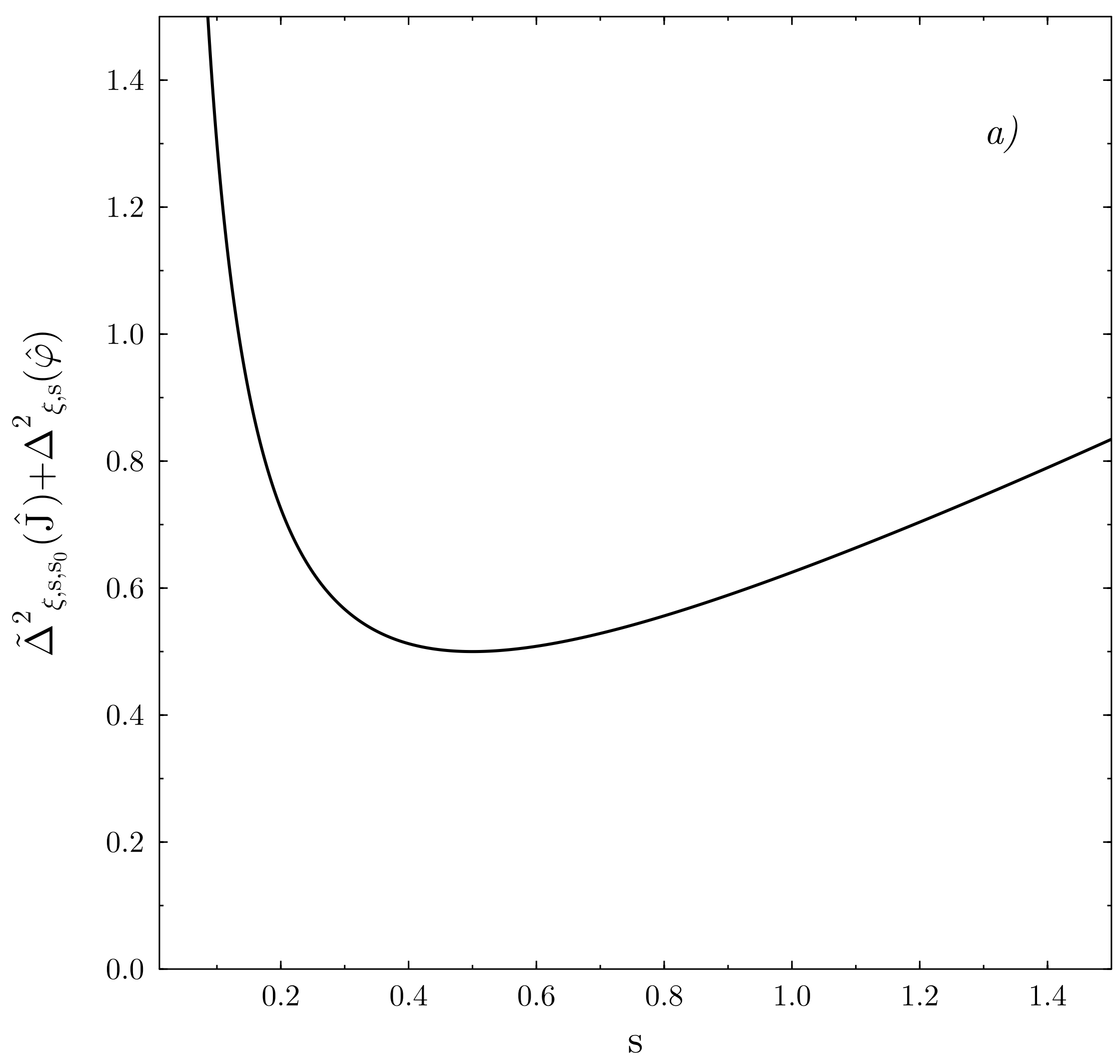
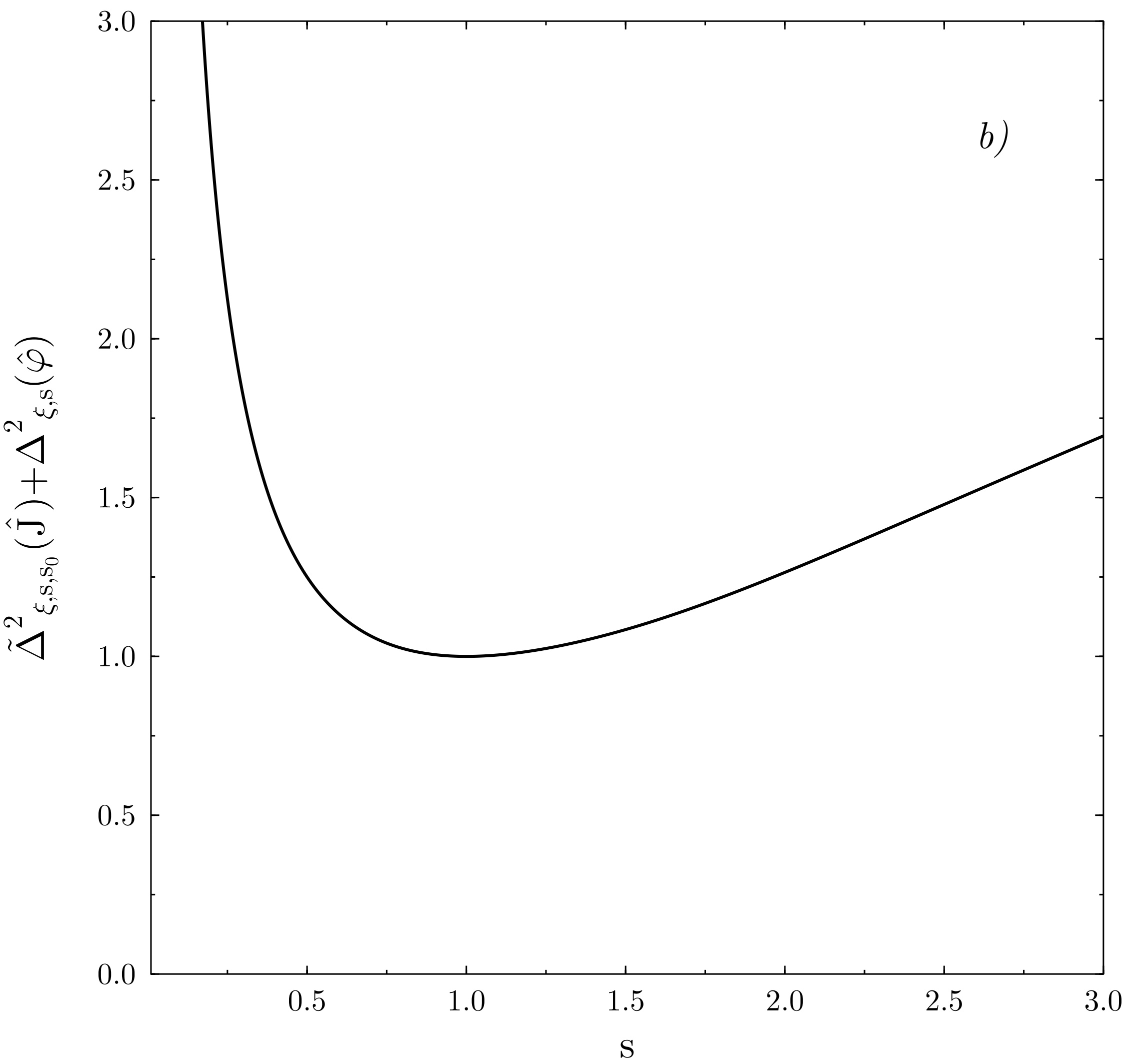
Figure captions

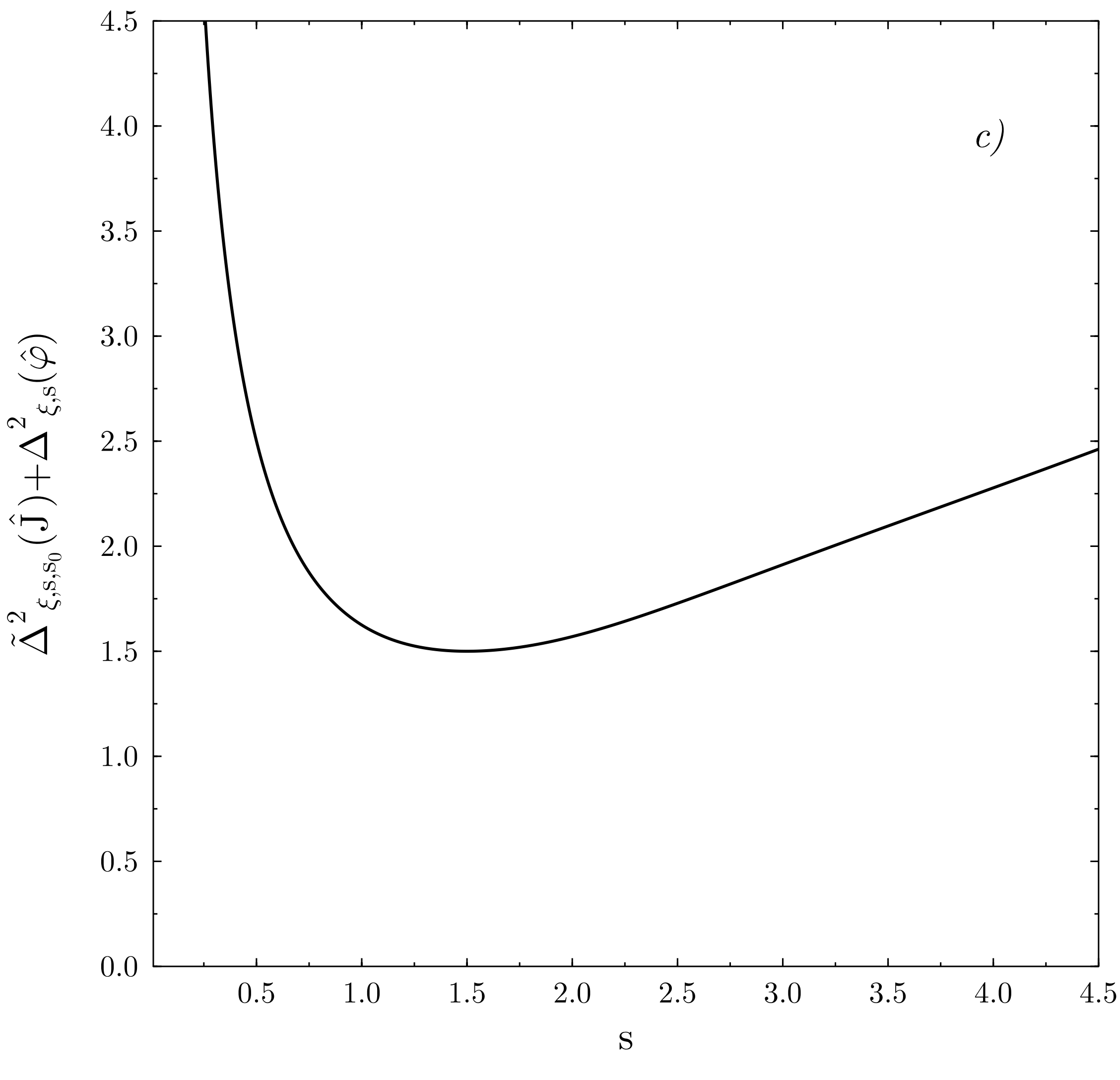
Figure 1. The plot of $\widetilde{\Delta}^2_{\xi,s,s_0}(\hat{J}) + \Delta^2_{\xi,s}(\hat{\varphi})$ (compare (4.15)), for a) $s_0 = .5$, b) $s_0 = 1$, and c) $s_0 = 1.5$, where $\widetilde{\Delta}^2_{\xi,s,s_0}(\hat{J}) = \frac{1}{4}\ln\left(\langle e^{-2s_0\hat{J}}\rangle_\phi \langle e^{2s_0\hat{J}}\rangle_\phi\right)$ and the expectation values from the argument of the logarithm are determined from (3.1) with $\lambda = \pm s_0$ and $l = 1$; the uncertainty of the angle given by (4.13), (3.6) and (4.10) is $s/2$. In accordance with (4.15) and (4.14) the coordinates of the minima are (s_0, s_0) (see also Fig. 2). We point out that the case b) with $s_0 = 1$ refers to the coherent states. More precisely, we have $\widetilde{\Delta}^2_{\xi,s,s_0}(\hat{J}) + \Delta^2_{\xi,s}(\hat{\varphi}) \equiv \Delta^2_{\xi,s}(\hat{J}) + \Delta^2_{\xi,s}(\hat{\varphi})$.

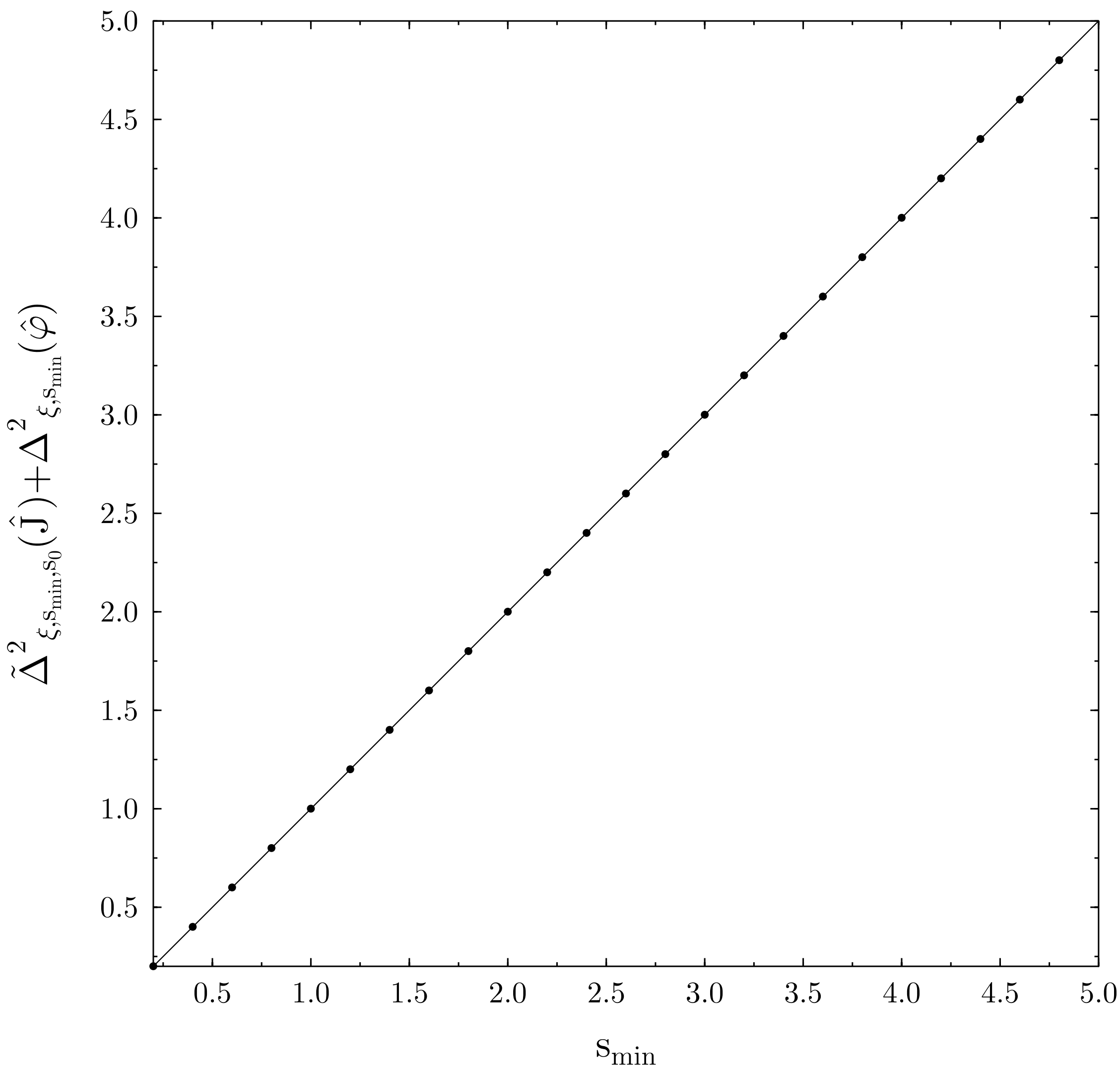
Figure 2. The plot of minima of the function from Fig. 1. As expected in view of (4.15) and (4.14) $s_{\rm min} = s_0$, and $\widetilde{\Delta}^2_{\xi,s_{\rm min},s_0}(\hat{J}) + \Delta^2_{\xi,s_{\rm min}}(\hat{\varphi}) = s_0$.


a)
$\tilde{\Delta}^2_{\xi,s,s_0}(\hat{J})+\Delta^2_{\xi,s}(\hat{\varphi})$
s
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
0.2
0.4
0.6
0.8
1.0
1.2
1.4



b)
$\tilde{\Delta}^2_{\xi,s,s_0}(\hat{J})+\Delta^2_{\xi,s}(\hat{\varphi})$
s
0.0
0.5
1.0
1.5
2.0
2.5
3.0


c)
$\tilde{\Delta}^2_{\xi,s,s_0}(\hat{J})+\Delta^2_{\xi,s}(\hat{\varphi})$
s
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5



5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
$\tilde{\Delta}^2_{\xi,s_{\min},s_0}(\hat{J})+\Delta^2_{\xi,s_{\min}}(\hat{\varphi})$
0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0 4.5 5.0
$s_{\min}$